

Two-loop Gell-Mann–Low function of N=1 supersymmetric Yang-Mills theory, regularized by higher covariant derivatives.

A.B.Pimenov *, K.V.Stepanyantz†

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*Moscow State University, physical faculty,
department of theoretical physics.
119992, Moscow, Russia*

Abstract

Two-loop Gell-Mann–Low function is calculated for N=1 supersymmetric Yang–Mills theory, regularized by higher covariant derivatives. The integrals, which define it, are shown to be reduced to total derivatives and can be easily calculated analytically.

1 Introduction.

It is well known that supersymmetry essentially improves the ultraviolet behavior of a theory. For example, even in theories with unextended supersymmetry, it is possible to propose the form of the β -function exactly to all orders of the perturbation theory. This proposal was first made in Ref. [1] as a result of investigating instanton contributions structure. For the $N = 1$ supersymmetric Yang-Mills theory, this β -function, called the exact Novikov–Shifman–Vainshtein–Zakharov (NSVZ) β -function is

$$\beta(\alpha) = -\frac{3C_2\alpha^2}{2\pi(1 - C_2\alpha/2\pi)}. \quad (1)$$

With the dimensional reduction, this expression coincides with the result of explicit calculating the function

$$b(\alpha) = \frac{d\alpha}{d\ln\mu}, \quad (2)$$

up to the four-loop approximation [2, 3, 4, 5], if a special choice of the renormalization prescription is used. Here α is a renormalized coupling constant, and μ is a normalization

*E-mail: pimenov.phys.msu@mail.ru

†E-mail: stepan@phys.msu.ru

point. However, function (2) is scheme dependent. The physical β -function is obtained from it only if the generating functional does not depend on the normalization point, and some special boundary conditions, which involve knowing finite parts of Green functions, are imposed. However, as a rule, the divergent part of the effective action in \overline{MS} scheme is only calculated with the dimensional reduction. We also note that the two-loop β -function was also calculated with the differential renormalization [6].

Investigation of the $N = 1$ supersymmetric electrodynamics (up to the four-loop approximation) [7, 8, 9, 10] shows that the exact NSVZ β -function coincides with the Gell-Mann-Low function.

This function is scheme independent, and can be calculated with an arbitrary regularization. However, using the higher derivative regularization [11, 12] is the most convenient. That matter is that with the higher derivative regularization all integrands, appearing in calculating the Gell-Mann-Low function, are total derivatives. It was first noted in Ref. [9]. Therefore, in fact, these integrals can be easily taken. In the electrodynamics this can be partially explained by a method, based on substituting solutions of Ward or Slavnov-Taylor identities into the Schwinger-Dyson equations [13]. However, for a complete proof it is necessary to propose existence of a new identity for the Green functions, which origin is so far unclear [14]. A similar identity [15] also appears in calculating a contribution of the matter superfields in a non-Abelian theory. It exists because integrals, defining the two-point Green function are reduced to total derivatives. Nevertheless, this statement was not yet verified in the non-Abelian case. And, in particular, the two-loop calculation of the Gell-Mann-Low function with the higher covariant derivatives (which is made in this paper) allows elucidating whether similar fact takes place in the non-Abelian case.

We note that using the higher covariant derivative regularization in non-Abelian theories is technically complicated. That is why such a regularization was applied only once, for the one-loop calculation in the (non-supersymmetric) Yang-Mills theory [16]. Taking into account comments, made in subsequent papers [17, 18, 19], the result of the calculation coincided with the the standard expression for the one-loop β -function (although in original paper [16] the authors affirm that it is not so). As we already mentioned, a purpose of this paper is calculating the Gell-Mann-Low function for the $N = 1$ supersymmetric Yang-Mills theory with the higher derivative regularization in the two-loop approximation. This function is defined as follows. Due to the Slavnov-Taylor identities a contribution to the renormalized effective action, corresponding to the two-point Green function of the gauge field, obtained using the background field method, is

$$-\frac{1}{8\pi}\text{tr}\int d^4\theta\frac{d^4p}{(2\pi)^4}\mathbf{V}(-p)\partial^2\Pi_{1/2}\mathbf{V}(p)d^{-1}(\alpha,\mu/p), \quad (3)$$

where \mathbf{V} is the background field, and $\partial^2\Pi_{1/2}$ is a supersymmetric transverse projection operator. Then the Gell-Mann-Low function is defined by

$$\beta\left(d(\alpha,\mu/p)\right)=\frac{\partial}{\partial\ln p}d(\alpha,\mu/p). \quad (4)$$

This paper is organized as follows.

In Sec. 2 we recall basic information about the $N = 1$ supersymmetric Yang-Mills theory, the background field method, and the higher derivatives regularization. Calculating a two-loop contribution to the Gell-Mann-Low function is described in Sec. 3.

Diagrams with counterterms insertions are calculated in Sec. 4 exactly to all orders of the perturbation theory. A brief discussion of the results is given in the conclusion.

2 $N = 1$ supersymmetric Yang-Mills theory, background field method and higher derivative regularization

The $N = 1$ supersymmetric Yang-Mills theory in the superspace is described by the action

$$S = \frac{1}{2e^2} \text{Re tr} \int d^4x d^2\theta W_a C^{ab} W_b. \quad (5)$$

Here the superfield W_a is a supersymmetric analogue of the gauge field stress tensor. It is defined by

$$W_a = \frac{1}{32} \bar{D}(1 - \gamma_5) D \left[e^{-2V} (1 + \gamma_5) D_a e^{2V} \right], \quad (6)$$

where V is a real scalar superfield, which contains the gauge field A_μ as a component, and

$$D = \frac{\partial}{\partial \theta} - i\gamma^\mu \theta \partial_\mu \quad (7)$$

is a supersymmetric covariant derivative. In our notation, the gauge superfield V is decomposed with respect to the generators of a gauge group T^a as $V = e V^a T^a$, where e is a coupling constant. The generators are normalized by the condition

$$\text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab}. \quad (8)$$

Action (5) is invariant under the gauge transformations

$$e^{2V} \rightarrow e^{i\Lambda^+} e^{2V} e^{-i\Lambda}, \quad (9)$$

where Λ is a chiral superfield.

For quantization of this model it is convenient to use the background field method. The matter is that the background field method allows calculating the effective action without manifest breaking of the gauge invariance. In the supersymmetric case it can be formulated as follows [20, 21]: Let us make a substitution

$$e^{2V} \rightarrow e^{2V'} \equiv e^{\mathbf{\Omega}^+} e^{2V} e^{\mathbf{\Omega}}, \quad (10)$$

in action (5), where $\mathbf{\Omega}$ is a background scalar superfield. Expression for V' is a complicated nonlinear function of V , $\mathbf{\Omega}$, and $\mathbf{\Omega}^+$. We do not interested in explicit form of this function:

$$V' = V'[V, \mathbf{\Omega}]. \quad (11)$$

(For brevity of notation we will not explicitly write the dependence on Ω^+ here and below.) The obtained theory will be invariant under the background gauge transformations

$$V \rightarrow e^{iK} V e^{-iK}; \quad e^\Omega \rightarrow e^{iK} e^\Omega e^{-i\Lambda}; \quad e^{\Omega^+} \rightarrow e^{i\Lambda^+} e^{\Omega^+} e^{-iK}, \quad (12)$$

where K is a real superfield, and Λ is a chiral superfield.

Let us construct the background chiral covariant derivatives

$$\mathbf{D} \equiv e^{-\Omega^+} \frac{1}{2} (1 + \gamma_5) D e^{\Omega^+}; \quad \bar{\mathbf{D}} \equiv e^\Omega \frac{1}{2} (1 - \gamma_5) D e^{-\Omega}. \quad (13)$$

Acting on a field X , which is transformed as $X \rightarrow e^{iK} X$, these covariant derivatives are transformed in the same way. It is also possible to define the covariant derivative with a Lorentz index

$$\mathbf{D}_\mu \equiv -\frac{i}{4} (C\gamma^\mu)^{ab} \{ \mathbf{D}_a, \bar{\mathbf{D}}_b \}, \quad (14)$$

which will have the same property. It is easy to see that after substitution (10) action (5) will be

$$S = \frac{1}{2e^2} \text{tr Re} \int d^4x d^4\theta \mathbf{W}^a \mathbf{W}_a - \frac{1}{64e^2} \text{tr Re} \int d^4x d^4\theta \left[16 \left(e^{-2V} \mathbf{D}^a e^{2V} \right) \mathbf{W}_a + \right. \\ \left. + \left(e^{-2V} \mathbf{D}^a e^{2V} \right) \bar{\mathbf{D}}^2 \left(e^{-2V} \mathbf{D}_a e^{2V} \right) \right], \quad (15)$$

where

$$\mathbf{W}_a = \frac{1}{32} e^\Omega \bar{\mathbf{D}} (1 - \gamma_5) D \left(e^{-\Omega} e^{-\Omega^+} (1 + \gamma_5) D_a e^{\Omega^+} e^\Omega \right) e^{-\Omega}, \quad (16)$$

and the notation

$$\begin{aligned} \mathbf{D}^2 &\equiv \frac{1}{2} \bar{\mathbf{D}} (1 + \gamma_5) \mathbf{D}; & \bar{\mathbf{D}}^2 &\equiv \frac{1}{2} \bar{\mathbf{D}} (1 - \gamma_5) \mathbf{D}; \\ \mathbf{D}^a &\equiv \left[\frac{1}{2} \bar{\mathbf{D}} (1 + \gamma_5) \right]^a; & \mathbf{D}_a &\equiv \left[\frac{1}{2} (1 + \gamma_5) \mathbf{D} \right]_a; \\ \bar{\mathbf{D}}^a &\equiv \left[\frac{1}{2} \bar{\mathbf{D}} (1 - \gamma_5) \right]^a; & \bar{\mathbf{D}}_a &\equiv \left[\frac{1}{2} (1 - \gamma_5) \mathbf{D} \right]_a \end{aligned} \quad (17)$$

is used. Action of the covariant derivatives on the field V in the adjoint representation is defined by the standard way.

We note that action (15) is also invariant under the quantum transformations

$$e^{2V} \rightarrow e^{-\lambda^+} e^{2V} e^{-\lambda}; \quad \Omega \rightarrow \Omega; \quad \Omega^+ \rightarrow \Omega^+ \quad (18)$$

where λ is a background chiral superfield, which satisfies the condition

$$\bar{\mathbf{D}} \lambda = 0. \quad (19)$$

Such a superfield can be presented as $\lambda = e^\Omega \Lambda e^{-\Omega}$, where Λ is a usual chiral superfield.

It is convenient to choose a regularization and gauge fixing so that invariance (12) will be unbroken. First, we fix the gauge by adding the following terms

$$S_{gf} = -\frac{1}{32e^2} \text{tr} \int d^4x d^4\theta \left(V \mathbf{D}^2 \bar{\mathbf{D}}^2 V + V \bar{\mathbf{D}}^2 \mathbf{D}^2 V \right) \quad (20)$$

to the action. In this case terms quadratic in the superfield V will have the simplest form:

$$\frac{1}{2e^2} \text{tr} \text{Re} \int d^4x d^4\theta V \mathbf{D}_\mu^2 V. \quad (21)$$

The corresponding action for the Faddeev–Popov ghosts S_c is written as

$$S_c = i \text{tr} \int d^4x d^4\theta \left\{ (\bar{c} + \bar{c}^+) V \left[(c + c^+) + \text{cth} V (c - c^+) \right] \right\}. \quad (22)$$

The superfield V in this expression is decomposed with respect to the generators of the adjoint representation of the gauge group, and the fields c and \bar{c} are the anticommuting background chiral fields.

Moreover [20], the quantization procedure also requires adding the action for the Nielsen–Kallosh ghosts

$$S_B = \frac{1}{4e^2} \text{tr} \int d^4x d^4\theta B^+ e^{\Omega^+} e^{\Omega} B, \quad (23)$$

where B is an anticommuting chiral superfield, and the background field should be decomposed with respect to the generators of the adjoint representation of the gauge group. Because the fields B and B^+ do not interact with the quantum gauge field, they contribute only to the one-loop (including subtraction) diagrams. It is important to note that the factor $1/e^2$ in action (23) is the same as in action for the gauge fixing terms (20).

The gauge fixing breaks the invariance of the action under the quantum gauge transformations (18), but there is a remaining invariance under the BRST-transformations. The BRST-invariance leads to the Slavnov–Taylor identities, which relate vertex functions of the quantum gauge field and ghosts. However, all these fields are present only in loops. Later we will introduce such a regularization that the BRST-invariance is broken, but background invariance (12) remains unbroken. Then a result of the calculation is surely gauge invariant (in a sense of the invariance under the background gauge transformations). A more complicated question is if it is possible to construct the renormalized effective action, which satisfies the Slavnov–Taylor identities. A possibility of using non-invariant regularizations was investigated in Refs. [22, 23, 24, 25]. According to these papers, to construct the effective action, satisfying the Slavnov–Taylor identities, it is necessary to use a special subtraction scheme, cancelling noninvariant terms in each order of the perturbation theory. With the background field method this scheme is slightly simplified, because the background gauge invariance guarantees, for example, the transversality of the two-point Green function for the gauge field. Nevertheless, as earlier, there are additional subtractions in the Green functions, containing the ghost fields.

However, it is necessary to clear up if using this regularization affects the result of calculating the Gell-Mann–Low function, which is investigated in this paper. To answer

this question, as a starting point we will use the following statement: If we fix a normalization point $\mu \ll \Lambda$ and impose in this point the boundary condition for the renormalized two-point Green function $d(p/\mu = 1)$, then the two-point Green function is uniquely determined and does not depend on both a way of renormalization and a regularization. For example, if two different regularizations are used, then

$$d_1\left(\alpha_1(\mu), \frac{p}{\mu}\right) = d_2\left(\alpha_2(\mu), \frac{p}{\mu}\right), \quad (24)$$

where $\alpha_i(\mu)$ and d_i are the renormalized coupling constants at the scale μ , and the renormalized two-point Green functions, obtained in the first and in the second regularization respectively. Setting $p = \mu$ in Eq. (24) it is possible to find the dependence $\alpha_1(\alpha_2)$. Therefore, two different regularizations differ in a finite renormalization of the coupling constant. We note that such a renormalization can be gauge dependent and causes the gauge dependence of the effective action divergent part in a sufficiently large order of the perturbation theory. However, the Gell-Mann–Low function, which we will calculate below in this paper, does not depend on such finite renormalization, because (setting $x \equiv \ln p/\mu$)

$$\beta_1\left(d_1(\alpha_1, x)\right) = \frac{\partial}{\partial x}d_1(\alpha_1, x) = \frac{\partial}{\partial x}d_2(\alpha_2, x) = \beta_2\left(d_2(\alpha_2, x)\right) = \beta_2\left(d_1(\alpha_1, x)\right). \quad (25)$$

Therefore, the Gell-Mann–Low function does not depend on a regularization. In particular, a regularization can break the BRST-invariance, provided the renormalized action is obtained by subtractions, restoring the Slavnov–Taylor identities. The Gell-Mann–Low function is gauge independent, because the dependence of the RHS on the gauge is factorized to the gauge dependence of the β -function argument. Therefore, using a regularization, breaking the BRST-invariance, is possible. In particular, we will add the term

$$S_\Lambda = \frac{1}{2e^2} \text{tr Re} \int d^4x d^4\theta V \frac{(\mathbf{D}_\mu^2)^{n+1}}{\Lambda^{2n}} V. \quad (26)$$

to action (15).

Proposed way of the regularization and gauge fixing preserves both invariance under the supersymmetry transformations and the invariance under transformations (12). As a consequence, the effective action, calculated with the background field method, will be invariant under both supersymmetry and gauge transformations.

We note that the regularization, described here, is different from a method, proposed in Ref. [26]. They differ in form of the term with higher covariant derivatives. In the method, considered here, it breaks the BRST-invariance, but terms, quadratic in the quantum superfield V , are simpler. This simplifies calculations in a certain degree, but all typical features of the higher derivative regularization are the same in the both cases. However, we should keep in mind that it is necessary to use a special subtraction scheme, because the higher derivative term breaks the BRST-invariance. This scheme cancels noninvariant terms, and ensures that the Slavnov–Taylor identities are satisfied in each order of the perturbation theory. It will be discussed in Sec. 4 in more details.

Let us construct the generating functional as follows:

$$Z[J, \mathbf{\Omega}] = \int D\mu \exp \left\{ iS + iS_\Lambda + iS_{gf} + iS_{gh} + \right. \\ \left. + i \int d^4x d^4\theta \left(J + J[\mathbf{\Omega}] \right) \left(V'[V, \mathbf{\Omega}] - \mathbf{V} \right) \right\}, \quad (27)$$

where the superfield \mathbf{V} is given by

$$e^{2\mathbf{V}} \equiv e^{\mathbf{\Omega}^+} e^{\mathbf{\Omega}}, \quad (28)$$

and $J[\mathbf{\Omega}]$ is a so far undefined functional. A reason of its introducing will be clear later. S_{gf} denotes gauge fixing terms (20), and $S_{gh} = S_c + S_B$ is the corresponding action for the Faddeev–Popov and Nielsen–Kallosh ghosts. The functional integration measure is written as

$$D\mu = DV D\bar{c} Dc DB. \quad (29)$$

We will assume that the coupling constant e is replaced by the bare coupling constant e_0 in all expressions.

In order to understand how generating functional (27) is related with the ordinary effective action, we perform the substitution $V \rightarrow V'$. Then we obtain

$$Z[J, \mathbf{\Omega}] = \exp \left\{ -i \int d^4x d^4\theta \left(J + J[\mathbf{\Omega}] \right) \mathbf{V} \right\} Z_0 \left[J + J[\mathbf{\Omega}], \mathbf{\Omega} \right], \quad (30)$$

where

$$Z_0[J, \mathbf{\Omega}] = \int D\mu \exp \left\{ iS + iS_\Lambda + iS_{gf} + iS_{gh} + i \int d^4x d^4\theta JV \right\}. \quad (31)$$

If the dependence of S , S_Λ , S_{gf} , and S_{gh} on the arguments V , $\mathbf{\Omega}$, and $\mathbf{\Omega}^+$ were factorized into the dependence on the variable V' , Z_0 would not depend on $\mathbf{\Omega}$ and $\mathbf{\Omega}^+$ and would coincide with the ordinary generating functional. This really takes place for action (5). However, in the term with the higher derivatives, in the gauge fixing terms, and in the ghost Lagrangian such factorization does not occur. Therefore, Z_0 actually differs from the ordinary generating functional.

Using the functional $Z[J, \mathbf{\Omega}, j]$ it is possible to construct the generating functional for the connected Green functions

$$W[J, \mathbf{\Omega}] = -i \ln Z[J, \mathbf{\Omega}] = - \int d^4x d^4\theta \left(J + J[\mathbf{\Omega}] \right) \mathbf{V} + W_0 \left[J + J[\mathbf{\Omega}], \mathbf{\Omega} \right]. \quad (32)$$

Also it is possible to construct the corresponding effective action

$$\Gamma[V, \mathbf{\Omega}] = - \int d^4x d^4\theta \left(J\mathbf{V} + J[\mathbf{\Omega}]\mathbf{V} \right) + W_0 \left[J + J[\mathbf{\Omega}], \mathbf{\Omega} \right] - \int d^4x d^4\theta JV, \quad (33)$$

where the sources should be expressed in terms of fields using the equation

$$V = \frac{\delta}{\delta J} W[J, \Omega] = -\mathbf{V} + \frac{\delta}{\delta J} W_0[J + J[\Omega], \Omega]. \quad (34)$$

Substituting this expression into Eq. (33), we write the effective action as

$$\Gamma[V, \Omega] = W_0[J + J[\Omega], \Omega] - \int d^4x d^4\theta \left(J[\Omega] \mathbf{V} + J \frac{\delta}{\delta J} W_0[J + J[\Omega], \Omega] \right). \quad (35)$$

Let us now set $V = 0$, so that

$$\mathbf{V} = \frac{\delta}{\delta J} W_0[J + J[\Omega], \Omega]. \quad (36)$$

We also take into account that the invariance under background gauge transformations (12) essentially restricts the form of the effective action. If the quantum field V in the effective action is set to 0, then the superfield K will be only in the gauge transformation law of the fields Ω and Ω^+ , the only invariant combination being expression (28). (It is invariant in a sense that the corresponding transformation law does not contain the superfield K .) This means that in the final expression for the effective action we can set

$$\Omega = \Omega^+ = \mathbf{V}. \quad (37)$$

In this case the effective action is

$$\Gamma[0, \mathbf{V}] = W_0[J + J[\mathbf{V}], \mathbf{V}] - \int d^4x d^4\theta \left(J + J[\mathbf{V}] \right) \frac{\delta}{\delta J} W_0[J + J[\mathbf{V}], \mathbf{V}]. \quad (38)$$

Note that this expression does not depend on form of the functional $J[\Omega]$. In particular, it can be chosen to cancel terms linear in the field V in Eq. (27). Such a choice will be very convenient below.

If the gauge fixing terms, ghosts, and the terms with higher derivatives depended only on V' , expression (38) would coincide with the ordinary effective action. However, as we already mentioned above, the dependence on V , Ω , and Ω^+ is not factorized into the dependence on V' with the proposed method of regularization and gauge fixing. According to Ref. [27, 28], the invariant charge (and, therefore, the Gell-Mann–Low function) is gauge independent, and the dependence of the effective action on gauge can be eliminated by renormalization of the wave functions of the gauge field, ghosts, and matter fields. Therefore, for calculating the Gell-Mann–Low function we may use the background gauge described above. We note that if this gauge is used, the renormalization constant of the gauge field A_μ is 1 due to the invariance of the action under transformations (12). We note that, as we already mentioned above, using a regularization, breaking the BRST-invariance does not change the Gell-Mann–Low function.

Nevertheless, generating functional (27) is not yet completely constructed. The matter is that adding the term with higher derivatives does not remove divergences from one-loop diagrams. To regularize them, it is necessary to insert the Pauli-Villars determinants in the generating functional [12]. The Pauli-Villars fields should be introduced for the quantum gauge field and ghosts (including the Nielsen–Kallosh ghosts). Constructing them we will at once use condition (37).

So, we insert in the generating functional the factors

$$\prod_i \left(\det PV(V, \mathbf{V}, M_i) \right)^{c_i}, \quad (39)$$

in which the Pauli-Villars determinants are defined by

$$\left(\det PV(V, \mathbf{V}, M) \right)^{-1} = \int DV_{PV} D\bar{c}_{PV} Dc_{PV} DB_{PV} \exp \left(iS_{PV} \right), \quad (40)$$

where the action for the Pauli-Villars fields is

$$\begin{aligned} S_{PV} \equiv & \text{tr Re} \int d^4x d^4\theta V_{PV} \left[\frac{1}{2e_0^2} \mathbf{D}_\mu^2 \left(1 + \frac{\mathbf{D}_\mu^{2n}}{\Lambda^{2n}} \right) - \frac{1}{e_0^2} \mathbf{W}^a \mathbf{D}_a + \frac{1}{e^2} M_V^2 \right] V_{PV} + \\ & + \frac{1}{4} \text{tr} \int d^4x d^4\theta (\bar{c}_{PV} + \bar{c}_{PV}^+) V \left[(c_{PV} + c_{PV}^+) + \text{cth } V (c_{PV} - c_{PV}^+) \right] + \\ & + \left(\frac{1}{2} M_c \text{tr} \int d^4x d^2\theta \bar{c}_{PV} c_{PV} + \text{h.c.} \right) + \frac{1}{4e_0^2} \text{tr} \int d^4x d^4\theta B_{PV}^+ e^{2\mathbf{V}} B_{PV} + \\ & + \text{tr} \left(\frac{1}{2e^2} \int d^4x d^2\theta M_B B_{PV}^2 + \text{h.c.} \right). \end{aligned} \quad (41)$$

The Grassmanian parity of the Pauli-Villars fields is opposite to the Grassmanian parity of usual fields, corresponding to them. The coefficients c_i in Eq. (39) satisfy conditions

$$\sum_i c_i = 1; \quad \sum_i c_i M_i^2 = 0. \quad (42)$$

Below, we assume that $M_i = a_i \Lambda$, where a_i are some constants. Inserting the Pauli-Villars determinants allows cancelling the remaining divergences in all one-loop diagrams, including diagrams containing counterterm insertions. (This is guaranteed because the masses of the gauge field and Nielsen-Kallosh ghosts are multiplied by the renormalized coupling constants, and the other terms are multiplied by the bare ones. This will be discussed later in more details.)

3 Two-loop calculation

The one-loop β -function, calculated with the background field method, is well-known [20]. Using the higher covariant derivative regularization does not essentially change the calculation, and its result [19]. Let us mention the typical features. The quantum superfield V does not contribute to the one-loop diagrams, because in the corresponding diagrams the number of the spinor derivatives D , acting on propagators, is less than 4. Really, a result of calculating any two-point diagram is proportional to

$$\delta_{xy}^8 \hat{P} \delta_{xy}^8, \quad (43)$$

where x and y are the points, to which the external lines are attached. The result is not 0 only if the operator \hat{P} contains 4 spinor derivatives. However, two vertexes can contain no

more than 2 spinor derivatives, and propagators of the gauge field do not contain spinor derivatives at all. Therefore, all one-loop two-point diagrams are automatically 0. The one-loop diagrams with the Pauli–Villars fields, corresponding to the gauge field, are 0 due to the same reason. Because the higher derivatives do not change a number of spinor derivatives in vertexes, the one-loop contribution of the quantum field is also 0 in the regularized theory.

Therefore, the one-loop two-point Green-function of the gauge field is completely determined by contributions of the Faddeev–Popov and Nielsen–Kallosh ghosts. With the regularization and gauge fixing, described above, the ghost Lagrangians do not depend on the presence of higher derivative terms. Due to anticommuting, the contributions of each ghost fields have opposite sign in comparison with the contribution of the chiral scalar superfield in the adjoint representation of the gauge group. Therefore, in the one-loop approximation the Gell-Mann–Low function is

$$\beta(\alpha) = -\frac{3C_2\alpha^2}{2\pi} + O(\alpha^3). \quad (44)$$

The effective action in the two-loop approximation is calculated by the standard way. It is contributed by diagrams, schematically presented in Fig. 1. Usual diagrams are obtained by attaching to them two external lines of the background gauge field by all possible ways. In Fig. 1 a propagator of the quantum field V is denoted by a wavy line, a propagator of the Faddeev–Popov ghosts by dashes, and a propagator of the Nielsen–Kallosh ghosts by dots. (We note that they contribute only in the one-loop approximation, because the Nielsen–Kallosh ghosts interact only with the background field.)

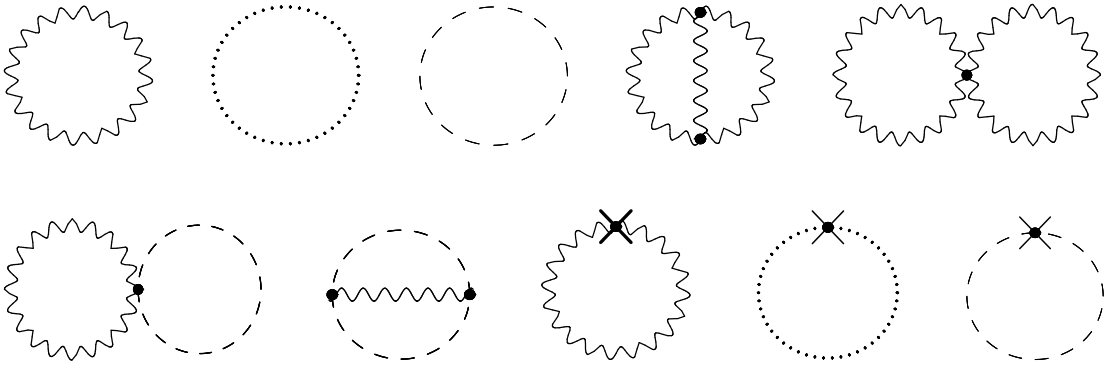


Figure 1: Diagrams, contributing to the two-loop β -function of the $N = 1$ supersymmetric Yang–Mills theory.

With the higher derivative regularization the propagator of the quantum field is

$$\frac{1}{q^2(1 + q^{2n}/\Lambda^{2n})} \quad (45)$$

(in the Euclidean space after the Weak rotation). Feynman rules for vertexes, containing two lines of the quantum field V , are also changed. In particular, a vertex with a single

line of the background superfield \mathbf{V} , which has the momentum p , (it is denoted by a bold wavy line) is

$$\text{wavy line} \sim \frac{1}{4}(2k+p)_\mu \bar{D}\gamma^\mu \gamma_5 D\mathbf{V} \left(1 + \frac{(k+p)^{2n+2} - k^{2n+2}}{\Lambda^{2n}((k+p)^2 - k^2)}\right), \quad (46)$$

and a vertex with two lines of the background superfield \mathbf{V} , which have momentums p and $-p$ is

$$\begin{aligned} \text{vertex} \sim & \left(4\mathbf{V}\partial^2\Pi_{1/2}\mathbf{V} + p^2\mathbf{V}^2\right) \left(1 + (n+1)\frac{k^{2n}}{\Lambda^{2n}}\right) + \frac{1}{\Lambda^{2n}} \left((2k+p)^2\mathbf{V}\partial^2\Pi_{1/2}\mathbf{V} + \right. \\ & \left. + \mathbf{V}^2((k+p)^2 - k^2)^2\right) \left(\frac{(k+p)^{2n+2} - k^{2n+2}}{((k+p)^2 - k^2)^2} - \frac{(n+1)k^{2n}}{(k+p)^2 - k^2}\right) - 4\mathbf{V}\partial^2\Pi_{1/2}\mathbf{V}. \quad (47) \end{aligned}$$

According to the performed calculations, the two-loop contribution of the Faddeev–Popov ghosts to the Gell-Mann–Low function is 0 that agrees, for example, with Ref. [29]. (The integrals, defining the two-point Green function, appeared to be some finite constants for the ghosts.)

As we already mentioned, the total two-loop contribution of the two-point diagrams to the effective action can be presented in form (3) due to the Slavnov–Taylor identity. To find the function d^{-1} up to an unessential constant, we differentiate it with respect to $\ln \Lambda$, and then set the external momentum p to 0. (This is possible due to using the higher covariant derivative regularization.) Later we will see that the result is a some finite constant d_2 :

$$\left.\frac{d}{d\ln\Lambda}d^{-1}(\alpha, \Lambda/p)\right|_{p=0} = d_2. \quad (48)$$

Therefore, the function d^{-1} depends on the momentum logarithmically

$$d^{-1}(\alpha, \Lambda/p) = d_2 \ln \frac{\Lambda}{p} + \text{const.} \quad (49)$$

Calculating explicitly two-loop diagrams, presented in Fig. 1 (so far without diagrams with counterterm insertions), differentiating the result with respect to $\ln \Lambda$, and, then, setting $p = 0$, we obtain (in the Euclidean space, after the Weak rotation)

$$\begin{aligned} d_2 = & 8\pi \cdot 6\pi \alpha_0 \frac{d}{d\ln\Lambda} \int \frac{d^4q}{(2\pi)^4} \left(q^2(1 + q^{2n}/\Lambda^{2n})\right)^{-1} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{d}{dk^2} \left\{ \left((q+k)^2 \times \right. \right. \\ & \left. \left. \times (1 + (q+k)^{2n}/\Lambda^{2n})\right)^{-1} \left[2(n+1)\left(1 + k^{2n}/\Lambda^{2n}\right)^{-1} - 2n\left(1 + k^{2n}/\Lambda^{2n}\right)^{-2} \right] \right\}. \quad (50) \end{aligned}$$

It is important to note that taking a limit $p \rightarrow 0$ is rather nontrivial, because the final result can contain infrared divergent terms, proportional to p or p^2 , or terms, proportional to p , but giving a finite contribution to d_2 . However, the calculation shows that all such

terms are cancelled. Moreover, the sum of diagrams appeared to be a total derivative with respect to the module of the loop momentum, so that the integral over d^4k , which is contained in Eq. (50), can be easily calculated. Really, in the four-dimensional spherical coordinates

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{d}{dk^2} f(k^2) = \frac{1}{16\pi^2} \left(f(k^2 = \infty) - f(k^2 = 0) \right). \quad (51)$$

All substitutions at the upper limit are 0 due to the higher derivative regularization, and only the substitution at the lower limit is nonzero. Using equations, presented above, we obtain

$$d_2 = -6\alpha_0 \frac{d}{d \ln \Lambda} \int \frac{d^4q}{(2\pi)^4} \left(q^2 (1 + q^{2n}/\Lambda^{2n}) \right)^{-2}. \quad (52)$$

This integral can be also easily calculated in the four-dimensional spherical coordinates:

$$d_2 = \frac{12\alpha_0}{\pi} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^4} q^2 \frac{d}{dq^2} (1 + q^{2n}/\Lambda^{2n})^{-2} = \frac{3\alpha_0}{4\pi^3} (1 + q^{2n}/\Lambda^{2n})^{-2} \Big|_0^\infty = -\frac{3\alpha_0}{4\pi^2}. \quad (53)$$

(We note that the result does not depend on the regularization parameter n .) Therefore, in the two-loop approximation

$$d^{-1}(\alpha_0, \Lambda/p) = \frac{1}{\alpha_0} - \frac{3C_2}{2\pi} \ln \frac{\Lambda}{p} - \frac{3\alpha_0 C_2^2}{(2\pi)^2} \ln \frac{\Lambda}{p} + O(\alpha_0^2). \quad (54)$$

Therefore, the Gell-Mann–Low function, defined by Eq. (4), in the two-loop approximation is

$$\beta(\alpha) = -\frac{3C_2\alpha^2}{2\pi} - \frac{3\alpha^3 C_2^2}{(2\pi)^2} + O(\alpha^4), \quad (55)$$

and coincides with the expansion of the exact NSVZ β -function in the considered order. We note that this result does not depend on a possible finite constant in Eq. (54).

4 Calculating diagrams with counterterms insertions

For calculating quantum corrections it is also necessary to take into account diagrams with counterterm insertions. Usually, adding counterterms is equivalent to splitting the bare coupling constant into the renormalized coupling constant and some infinite additional term. However, using noninvariant regularizations (and, in particular, the regularization, breaking the BRST-invariance, which is used here), it is also necessary to add counterterms, restoring the Slavnov–Taylor identities [22, 23] in each order of the perturbation theory. However, in the considered case the situation is slightly simplified. Really, the one-loop two-point Green function for the Faddeev–Popov ghosts is finite and does not depend on regularization. Interaction of ghosts with the background field is fixed by the background gauge invariance, which is unbroken in the considered regularization. Therefore, additional counterterms do not contribute to subtraction diagrams, containing

a loop of the Faddeev–Popov ghosts, in the two-loop approximation. Moreover, terms with the Faddeev–Popov ghosts do not evidently depend on whether bare or renormalized coupling constant is in the gauge fixing action. Therefore, their contributions do not also depend on a way of splitting the bare coupling constant into the renormalized one and counterterms.

Quantizing the theory we also write the bare coupling constant e_0 in the gauge fixing terms. Therefore, a part of the action, quadratic in the quantum field, is written as

$$\frac{1}{2e_0^2} \text{tr Re} \int d^4x d^4\theta V \left[\mathbf{D}_\mu^2 \left(1 + \frac{\mathbf{D}_\mu^{2n}}{\Lambda^{2n}} \right) + 2W^a \mathbf{D}_a \right] V. \quad (56)$$

Breaking the invariance under the BRST-transformations can lead to the necessity of adding counterterms proportional to

$$\text{tr} \int d^4x d^4\theta V \mathbf{D}_\mu^2 V. \quad (57)$$

(If the background field is 0, this follows from Refs. [24, 25]. Terms, containing the background field, can be restored due to the background gauge invariance.) But this means that all one-loop diagrams, *including diagrams with insertions both of the counterterms, appearing due to the renormalization of the coupling constant, and of the additional counterterms*, with a loop of the quantum field V , are 0, because they can contain no more than 2 spinor derivatives.

At last, let us consider diagrams, containing a loop of the Nielsen–Kallosh ghosts. Because the Nielsen–Kallosh ghosts exist only in the one-loop approximation, there are no additional counterterms, caused by the noninvariance of the regularization under the BRST-transformations, in these diagrams. However, the contribution of the counterterm diagrams is essential due to the renormalization of the coupling constant. Really, the coefficient in the action for the Nielsen–Kallosh ghosts should be the same as in the gauge fixing terms. Therefore, it must contain the bare coupling constant:

$$\frac{1}{4e_0^2} \text{tr} \int d^8x B^+ e^{2\mathbf{V}} B. \quad (58)$$

To regularize diagrams with counterterm insertions and a loop of Nielsen–Kallosh ghosts, the action for the corresponding Pauli–Villars fields should be written as

$$\text{tr} \int d^4x \left(\frac{1}{4e_0^2} \int d^4\theta B_{PV}^+ e^{2\mathbf{V}} B_{PV} + \frac{M_B}{2e^2} \int d^2\theta B_{PV}^2 + \frac{M_B}{2e^2} \int d^2\theta (B_{PV}^+)^2 \right), \quad (59)$$

where M_B is proportional to the regularization parameter Λ . Really, let us present a bare coupling constant as

$$\frac{1}{e_0^2} = \frac{1}{e^2} Z_3, \quad (60)$$

where e is the renormalized coupling constant, and Z_3 is the renormalization constant. Then, expanding the Pauli–Villars determinant for the Nielsen–Kallosh ghosts in powers of $Z_3 - 1$, we obtain terms, regularizing diagrams with insertions of counterterms.

However, due to inserting this determinant the generating functional starts to depend on the normalization point at the fixed bare coupling constant e_0 , because the renormalized coupling constant e depends on μ .

In the Abelian case calculating divergences for the action, similar to (59), was made, for example, in Ref. [7]. In the considered case it is also necessary to take into account a factor $-C_2/2$, which appears because the Nielsen–Kallosh ghosts are in the adjoint representation of a gauge group and anticommute. (There is only one matter superfield now, instead of 2 matter superfields in the Abelian case.) Moreover, the renormalization constant of the matter field Z should be substituted for the constant Z_3 . Taking into account this comments, the result of Ref. [7] can be formulated as follows. Contribution of the counterterm diagrams for the Nielsen–Kallosh ghosts to $1/d$ can be written as

$$\frac{C_2}{2\pi} \ln Z_3. \quad (61)$$

To find this contribution in the two-loop approximation we note that after the one-loop renormalization the renormalization constant will be

$$Z_3 = 1 + \frac{3C_2\alpha}{2\pi} \ln \frac{\Lambda}{\mu} + O(\alpha^2). \quad (62)$$

Therefore, the contribution of diagrams with counterterm insertions in the two-loop approximation is written as

$$\frac{3\alpha C_2^2}{(2\pi)^2} \ln \frac{\Lambda}{\mu}. \quad (63)$$

This contributions exactly cancels the two-loop divergence so that after the one-loop renormalization

$$d^{-1}(\alpha, \mu/p) = \frac{1}{\alpha} - \frac{3C_2}{2\pi} \ln \frac{\mu}{p} - \frac{3\alpha C_2^2}{(2\pi)^2} \ln \frac{\mu}{p} + O(\alpha^2). \quad (64)$$

For an arbitrary order of the perturbation theory it is reasonable to propose that the two-point Green function of the gauge field is given by

$$\frac{1}{d(\alpha, \mu/p)} = \frac{1}{\alpha_0} - \frac{1}{2\pi} C_2 \ln d(\alpha_0, \Lambda/p) + \frac{1}{2\pi} C_2 \ln Z_3(\alpha, \Lambda/\mu) - \frac{3}{2\pi} C_2 \ln \frac{\Lambda}{p}. \quad (65)$$

Really, it is easy to see that the exact NSVZ β -function is obtained by differentiating this equality with respect to $\ln p$, and the term, proportional to $\ln Z_3$ is obtained from contributions of diagrams with counterterm insertions. In the two-loop approximation this equation agrees with (54) if the contribution of diagrams with counterterm insertions is taken into account.

If Eq. (65) is true, then divergences exist only in the one-loop approximation. Really, because

$$\frac{1}{d(\alpha, \mu/p)} = \frac{1}{d(\alpha_0(\alpha, \Lambda/\mu), \Lambda/p)} Z_3(\alpha, \Lambda/\mu) \quad (66)$$

is finite, it is necessary to cancel only the one-loop divergence. For this purpose the bare coupling constant is presented as

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} + \frac{3}{2\pi} C_2 \ln \frac{\Lambda}{\mu}. \quad (67)$$

We note that presence of divergences only in the one-loop approximation in this case does not mean that the physical β -function has only the one-loop contribution. Really, the physical β -function is a derivative of the two-point Green function with respect to the logarithm of the momentum if proper boundary conditions are imposed. Such function, as we already saw, has corrections in all loops. A relation between the divergences and the physical β -function is broken due to the way of the regularization of diagrams with the insertions of counterterms, which leads to the dependence of the generating functional on the normalization point at the fixed bare coupling constant [30].

So, if Eq. (65) is valid, the Gell-Mann–Low function coincides with the exact NSVZ β -function, and divergences in the two-point Green function exist only in the one-loop approximation.

5 Conclusion

Investigation, made in this paper, shows that the higher covariant derivative regularization can be easily applied for calculating quantum corrections in the supersymmetric Yang–Mills theory. Its using allows differentiating with respect to the regularization parameter Λ and setting the external momentum to 0. As a result, it is possible to find the Gell-Mann–Low function, which in the considered approximation coincides with the expansion of the exact NSVZ β -function. (We note, once again, that the Gell-Mann–Low function does not depend on the choice of the renormalization scheme.) A very interesting feature of using the higher covariant derivative regularization in supersymmetric theories is a possibility of calculating all integrals analytically, because their sum is reduced to a total derivative. Exactly the same feature was noted in the Abelian case [9].

With the higher derivative regularization divergences in the two-point Green function appeared to exist only in the one-loop approximation. (However, the divergent part of the two-point Green function is not a physical quantity. A physical quantity is the Gell-Mann–Low function, which are contributed by all orders of the perturbation theory.) The obtained result to a considerable extent confirms conclusions of Ref. [31], where the authors proposed that the Wilson action S_W was renormalized only at the one-loop, and the effective action Γ had corrections in all loops. In our case the usual renormalized action plays a role of S_W . As for the electrodynamics, the Gell-Mann–Low function does not coincide with the function $b(\alpha)$, defined by the divergent part of the effective action, due to the rescaling anomaly [32], which leads to the dependence of the standardly defined generating functional on the normalization point.

We note that using the higher covariant derivative regularization can possibly allow deriving the expression for the exact NSVZ β -function by the straightforward summation of Feynman diagrams exactly to all orders of the perturbation theory, similar to the case of the supersymmetric electrodynamics. Now this work is in progress.

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